NORMAL FORMS OF TRIANGLES AND QUADRILATERALS UP TO SIMILARITY

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Abstract. In this paper the problem of finding a normal form of triangles and plane quadrilaterals up to similarity is considered. Several normal forms for triangles and a normal form for quadrilaterals of special case are described. Normal forms of simple plane objects such as triangles can be used in mathematics teaching.

Key words. normal forms, similarity, triangle

AMS subject classifications. 51M04, 97G50, 97D99.

1. Introduction. Many problems of classical Euclidean geometry explicitly or implicitly consider objects up to similarity. Understanding and using similarity is an important geometry competence feature for schoolchildren.

Recall that two geometric figures A and B are similar if B can be obtained from A after a finite composition of translations, rotations, reflections and dilations (homotheties). Similarity is an equivalence relation and thus, for example, the set of all triangles in a plane is partitioned into similarity equivalence classes which can be identifies with *similarity types* of triangles.

In many areas of mathematics objects are studied up to equivalence relations. Depending on situation and traditions this is done explicitly, implicitly or inadvertently. The problem of finding a distinguished (*canonical, normal*) representatives of equivalence classes of objects ir posed. Alternatively, it is the problem of mapping the quotient set injectively back to the original set. Let X be a set with an equivalence relation $\sim$ or, equivalently, $R \subseteq X \times X$, denote the equivalence class of $x \in X$ by $[x]$. Let $\pi : X \to X/R$ such that $\pi(x) = [x]$ be the canonical projection map. We call a map $\sigma : X/R \to X$ *normal object map* provided 1) σ is injective and 2) $\sigma \circ \pi = id_X$. For example, there are various normal forms of matrices, such as the Jordan normal form. See [4] for examples of normal forms in algebra and [3] for a related recent work. Normal objects are designed for educational, pure research (e.g. for classification) and applied reasons. Normal objects are constructed as objects of simple, minimalistic design, to show essential properties and parameters of original

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objects. Often it is easier to solve a problem for normal objects first and extend the solution to atbitrary objects afterwards. Normal objects which are initially designed for educational, pure research or problem solving purposes are also used to optimize computations.

In elementary Euclidean geometry normal map approach does not seem to be popular working with simple discrete objects such as triangles. This may be related to the traditional dominance of the synthetic geometry in school mathematics at the expense of the coordinate/analytic approach. We can pose the problem of introducing and using normal forms of triangles up to similarity. This means to describe a set S of mutually non-similar triangles such that any triangle in the plane would be similar to a triangle in S. We assume that Cartesian coordinates are introduced in the plane, S is designed using the Cartesian coordinates. For triangle we offer three normal forms based on side lengths. Using these normal forms the set of triangle similarity forms is bijectively mapped to a fixed plane domain bounded by lines and circles. For these forms two vertices are fixed and the third vertex belongs to this finite domain, we call them *the one vertex normal forms*. One vertex normal forms are also generalized for quadrilaterals. Another normal form for triangles is based on angles and circumscribed circles. For this form one constant vertex is fixed on the unit circle and two other variable vertices also belong to the unit circle, we call this form *the circle normal form*.

These normal forms may be useful in solving geometry problems involving similarity and teaching geometry. The paper may be useful for mathematics educators.

2. Main results.

2.1. Normal forms of triangles.

2.1.1. Notations. Consider $\mathbb{R}^2$ with a Cartesian system of coordinates (x, y) and center O. We think of classical triangles as being encoded by their vertices. Strictly speaking by the triangle $\triangle XYZ$ we mean the multiset $\{\{X, Y, Z\}\}$ of three points in $\mathbb{R}^2$ each point having multiplicity at most 2. A triangle is called degenerate if points lie on a line. Given $\triangle ABC$ we denote $\angle BAC = \alpha$, $\angle ABC = \beta$, $\angle ACB = \gamma$, $|BC| = a$, $|AC| = b$, $|AB| = b$. We exclude mutisets of type $\{\{XXX\}\}$.

We will use the following affine transformations of $\mathbb{R}^2$: 1) translations, 2) rotations, 3) reflections with respect to an axis, 4) dilations (given by the rule $(x, y) \to (cx, cy)$ for some $c \in \mathbb{R}\backslash\{0\}$). It is known that these transformations generate the *dilation group* of $\mathbb{R}^2$, denoted by some authors as $IG(2)$, see [2], [3]. Two triangles T_1 and T_2 as similar if there exists $g \in IG(2)$, such that $g(T_1) = T_2$ (as multisets). If triangles T_1 and T_2 are similar, we write $T_1 \sim T_2$ or $\triangle X_1Y_1Z_1 \sim \triangle X_2Y_2Z_2$.

The point (x_1, y_1) is lexicographically smaller than the point (x_2, y_2) and denoted as $(x_1, y_1) \prec (x_2, y_2)$ provided $(x_1 < x_2)$ or $(x_1 = x_2$ and $y_1 < y_2)$. The lexicographical order of points can be extended to lexicographical ordering of sequences of points: the sequence of points $[p_1, p_2]$ is lexicographically smaller than the sequence $[q_1, q_2]$ denoted by $[p_1, p_2] \prec [q_1, q_2]$ provided $(p_1 \prec q_1)$ or $(p_1 = q_1$ and $p_2 \prec q_2)$.

We use normal letters to denote fixed objects and *mathcal* letters to denote objects as function values.

2.1.2. The C-vertex normal form. A normal form can be obtained by transforming the longest side of the triangle into a unit interval of the x-axis. We call it *the C-normal form.*

In this subsection $A = (0, 0)$ and $B = (1, 0)$.

Definition 2.1.

Let $S_C \subseteq \mathbb{R}^2$ be the domain in the first quadrant bounded by the lines $y = 0$, $x = \frac{1}{2}$ and the circle $x^2 + y^2 = 1$, see Figure 1.

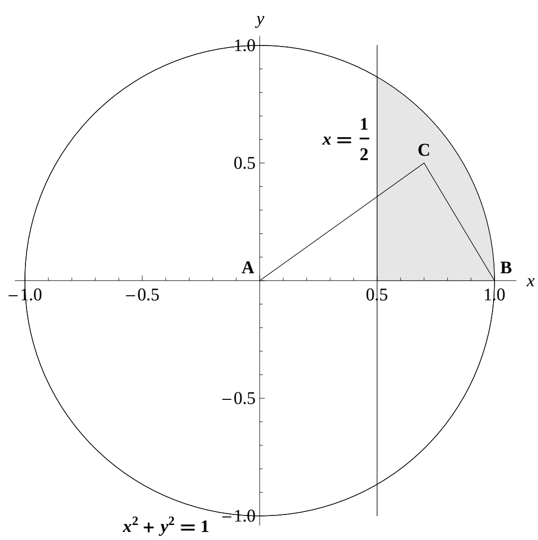

Fig.1. - the domain S_C.

In other terms, S_C is the set of solutions of the system of inequalities

$$\begin{cases} y \geq 0 \\ x \geq \frac{1}{2} \\ x^2 + y^2 \leq 1. \end{cases}$$

THEOREM 2.2. *Every triangle UVW (including degenerate triangles) in $\mathbb{R}^2$ is similar to a triangle ABC, where $A = (0,0)$, $B = (1,0)$ and $\mathcal{C} \in S_C$.*

Proof. Let $\triangle UVW$ has side lengths a, b, c satisfying $a \leq b \leq c$. Perform the following sequence of transformations:

1. translate and rotate the triangle so that the longest side is on the x-axis, one vertex has coordinates $(0,0)$ and another vertex has coordinates $(c,0)$, $c > 0$;
2. if the third vertex has negative y-coordinate, reflect the triangle with respect to the x-axis;
3. do the dilation with coefficient $\frac{1}{c}$, note that the vertices on the x-axis have coordinates $(0,0)$ and $(1,0)$, the third vertex has coordinates (x'_C, y'_C), where $x'^2_C + y'^2_C \leq 1$ and $(x'_C - 1)^2 + y'^2_C \leq 1$;
4. if $x'_C < \frac{1}{2}$, then reflect the triangle with respect to the line $x = \frac{1}{2}$, denote the third vertex by $\mathcal{C} = (x_C, y_C)$, by construction we have that $\mathcal{C} \in S_C$.

The image of the initial triangle $\triangle UVW$ is the triangle $AB\mathcal{C}$, where $\mathcal{C} \in S_C$. All transformations preserve similarity type therefore $\triangle UVW \sim \triangle AB\mathcal{C}$. ∎

THEOREM 2.3. *If $C_1 \in S_C$, $C_2 \in S_C$ and $C_1 \neq C_2$, then $\triangle ABC_1 \not\sim \triangle ABC_2$.*

Proof. If $\angle C_1AB = \angle C_2AB$ and $C_1 \neq C_2$, then $\angle C_1BA \neq \angle C_2BA$. By equality of angles for similar triangles it follows that $\triangle ABC_1 \not\sim \triangle ABC_2$.

Let $\angle C_1AB \neq \angle C_2AB$. The angle C_iAB is the smallest angle in $\triangle ABC_i$. By equality of angles for similar triangles it again follows that $\triangle ABC_1 \not\sim \triangle ABC_2$.
∎

DEFINITION 2.4. *A point $\mathcal{C} \in S_C$ such that $\triangle AB\mathcal{C} \sim \triangle UVW$ is called the* C-normal point of $\triangle UVW$.

DEFINITION 2.5. *The C-vertex normal form of $\triangle UVW$ is $\triangle AB\mathcal{C}$, where $\mathcal{C} \in S_C$ is the C-normal point of $\triangle UVW$.*

REMARK 2.6. *Denote by R_C the intersection of the circle $(x - \frac{1}{2})^2 + y^2 = (\frac{1}{2})^2$ and S_C. Points of R_C correspond to right angle triangles. Points below and above R_C correspond to, respectively, obtuse and acute triangles, see Figure 2.*

Points on the intersection of the line $x = \frac{1}{2}$ and S_C correspond to isosceles obtuse triangles. Points on the intersection of the circle $x^2 + y^2 = 1$ and S_C correspond to isosceles acute triangles. Points in the interior of S_C correspond to scalene triangles. The point $(\frac{1}{2}, \frac{\sqrt{3}}{2})$ corresponds to the equilateral triangle. Points on the intersection of the line $y = 0$ and S_C correspond to degenerate triangles. $\mathcal{C} = B$ for triangles having

side lengths $0, c, c$.

REMARK 2.7. *A similar normal form can be obtained reflecting* S_C *with respect to the line* $x = \frac{1}{2}$.

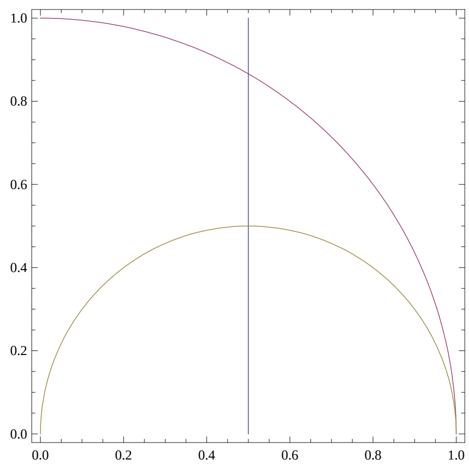

Fig.2. - the subdomains of S_C corresponding to obtuse and acute triangles.

2.1.3. The B-vertex normal form. Another normal form can be obtained by transforming the median length side (in the sense of ordering) of the triangle into a unit interval of the x-axis. By analogy it is called *the B-normal form*.

In this subsection $A = (0, 0)$ and $C = (1, 0)$.

DEFINITION 2.8.

Let $S_B \subseteq \mathbb{R}^2$ *be the domain in the first quadrant bounded by the line* $y = 0$ *and the circles* $x^2 + y^2 = 1$ *and* $(x - 1)^2 + y^2 = 1$*, see Figure 3.*

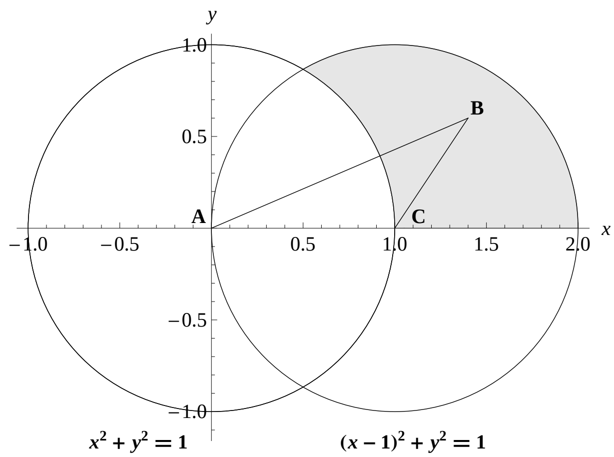

Fig.3. - the domain S_B.

In other terms, S_B is the set of solutions of the system of inequalities

$$\begin{cases} y \geq 0 \\ x^2 + y^2 \geq 1 \\ (x-1)^2 + y^2 \leq 1. \end{cases}$$

THEOREM 2.9. *Every triangle UVW (including degenerate triangles) in $\mathbb{R}^2$ is similar to a triangle $A\mathcal{B}C$, where $A = (0,0)$, $C = (1,0)$ and $\mathcal{B} \in S_B$.*

Proof. Let $\triangle UVW$ has side lengths a, b, c satisfying $a \leq b \leq c$. Perform the following sequence of transformations:

1. translate and rotate the triangle so that the side of length b is on the x-axis, one vertex has coordinates $(b, 0)$ and another vertex has coordinates $(b, 0)$, $b > 0$, the side of length c is incident to the vertex $(c, 0)$;
2. if the third vertex has negative y-coordinate, reflect the triangle with respect to x-axis;
3. do the dilation with coefficient $\frac{1}{b}$, note that the vertices on the x-axis have coordinates $(0, 0)$ and $(1, 0)$, at this point the third vertex $\mathcal{B}$ has coordinates (x'_B, y'_B), where $y'_B \geq 0$, $x'^2_B + y'^2_B \geq 1$ or $(x'_B - 1)^2 + y'^2_B \leq 1$;
4. if $x'_B < \frac{1}{2}$, reflect the triangle with respect to the line $x = \frac{1}{2}$, now the third vertex $\mathcal{B}$ has new coordinates (x_B, y_B), where $x_B \geq \frac{1}{2}$, $y_B \geq 0$, $(x_B - 1)^2 + y_B^2 \leq 1$.

The image of the initial triangle $\triangle UVW$ is the triangle $A\mathcal{B}C$, where $\mathcal{B} \in S_B$. All transformations preserve similarity type therefore $\triangle UVW \sim \triangle A\mathcal{B}C$. ☐

THEOREM 2.10. *If $B_1 = (x_i, y_i) \in S_B$, $B_2 = (x_2, y_2) \in S_B$ and $B_1 \neq B_2$, then*

$\triangle AB_1C \not\sim \triangle AB_2C$.

Proof. The angle $\angle B_iAC$ is the smallest angle in the triangle $\triangle AB_iC$.

If $\angle B_1AC \neq \angle B_2AC$, then, since these are smallest angles in the triangles, it follows that $\triangle AB_1C \not\sim \triangle AB_2C$.

If $\angle B_1AC = \angle B_2AC$ and $B_1 \neq B_2$, then $\angle AB_1C \neq \angle AB_2C$. $\angle AB_iC$ is the biggest angle in $\triangle AB_iC$, therefore $\angle AB_1C \neq \angle AB_2C$ implies $\triangle AB_1C \not\sim AB_2C$. $\square$

DEFINITION 2.11. *A point $\mathcal{B} \in S$ such that $\triangle A\mathcal{B}C \sim \triangle UVW$ is called the* B-normal point of $\triangle UVW$.

DEFINITION 2.12. *The B-vertex normal form of $\triangle UVW$ is $\triangle A\mathcal{B}C$, where $\mathcal{B} \in S$ is the B-normal point of $\triangle UVW$.*

REMARK 2.13. *Denote by R_B the intersection of the ray $x = 1$, $x \geq 0$ and S_B. Points of R_B correspond to right angle triangles. Points to the right and left of R_B correspond to, respectively, obtuse and acute triangles, see Figure 4.*

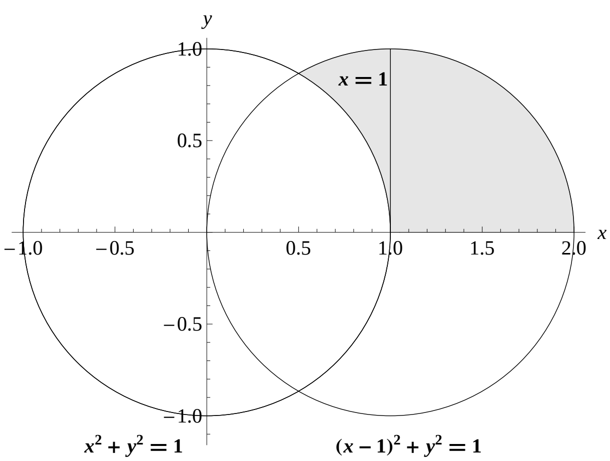

Fig.4. - the subdomains of S_B corresponding to obtuse and acute triangles.

Points on the intersection of the line $x^2 + y^2 = 1$ and S_B correspond to isosceles acute triangles. Points on the intersection of the circle $(x-1)^2 + y^2 = 1$ and S_B correspond to isosceles obtuse triangles. Points in the interior of S_B correspond to scalene triangles. The point $(\frac{1}{2}, \frac{\sqrt{3}}{2})$ corresponds to the equilateral triangle. Points on the intersection of the line $y = 0$ and S_B correspond to degenerate triangles. $\mathcal{B} = C$ for triangles having side lengths $0, c, c$.

2.1.4. ***A*-vertex normal form.** Finally a normal form can be obtained by transforming the shortest side of the triangle into a unit interval of the x-axis. By analogy it is called *the A-normal form*. In this case again two vertices on the x-axis are $(0,0)$ and $(1,0)$, the domain S_A of possible positions of the third vertex is unbounded.

In this subsection $B=(0,0)$ and $C=(1,0)$.

DEFINITION 2.14.

Let $S_A \subseteq \mathbb{R}^2$ be the unbounded domain in the first quadrant bounded by the lines $y=0$, $x=\frac{1}{2}$ and the circle $(x-1)^2+y^2=1$, see Figure 5.

In other terms, S_A is the set of solutions of the system of inequalities

$$\begin{cases} y \geq 0 \\ x \geq \frac{1}{2} \\ (x-1)^2+y^2 \geq 1. \end{cases}$$

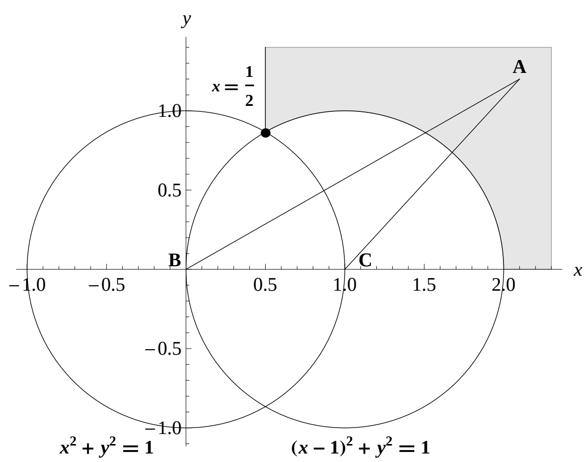

Fig.5. - the domain S_A.

THEOREM 2.15. *Every triangle UVW (including degenerate triangles but excluding the similarity type having side lengths $0,c,c$) in $\mathbb{R}^2$ is similar to a triangle $\mathcal{A}BC$, where $B=(0,0)$, $C=(1,0)$ and $\mathcal{A} \in S_A$.*

Proof. Let $\triangle UVW$ has side lengths a,b,c satisfying $a \leq b \leq c$. Perform the following sequence of transformations:

1. translate and rotate the triangle so that the side of length a is on the x-axis, one vertex has coordinates $(0,0)$ and another vertex has coordinates $(a,0)$, $a>0$, the side of length c is incident to the vertex $(0,0)$;

2. if the third vertex has negative y-coordinate, reflect the triangle with respect to the x-axis;
3. do the dilation with coefficient $\frac{1}{a}$, note that the vertices on the x-axis have coordinates $(0,0)$ and $(1,0)$;
4. if the third point has the x-coordinate less than $\frac{1}{2}$, reflect the triangle with respect to the line $x = \frac{1}{2}$, now the third vertex $\mathcal{A}$ has coordinates (x_0, y_0), where $x_0 \geq \frac{1}{2}$, $y_0 \geq 0$, $(x_0 - 1)^2 + y_0^2 \leq 1$.

The image of the initial triangle $\triangle UVW$ is the triangle $\mathcal{A}BC$, where $\mathcal{A} \in S_A$. All transformations preserve similarity type therefore $\triangle UVW \sim \triangle \mathcal{A}BC$. ∎

Theorem 2.16. *Let $B = (0,0)$, $C = (1,0)$. If $A_1 = (x_i, y_i) \in S_A$, $A_2 = (x_2, y_2) \in S_A$ and $A_1 \neq A_2$, then $\triangle A_1BC \not\sim \triangle A_2BC$.*

Proof. The angle $\angle BCA_i$ is the largest angle in the triangle $\triangle A_iBC$.

If $\angle BCA_1 \neq \angle BCA_2$, then since these are largest angles in the triangles it follows that $\triangle A_1BC \not\sim \triangle A_2BC$.

If $\angle BCA_1 = \angle BCA_2$ and $A_1 \neq A_2$, then $\angle BA_1C \neq \angle BA_2C$. BA_iC is the smallest angle in $\triangle A_iBC$, therefore $A_1BC \neq A_2BC$ implies $\triangle AB_1C \not\sim AB_2C$.
∎

Definition 2.17. *A point $\mathcal{A} \in S_A$ such that $\triangle \mathcal{A}BC \sim \triangle UVW$ is called the* A-normal point of $\triangle UVW$.

Definition 2.18. *The A-vertex normal form of $\triangle UVW$ is $\triangle \mathcal{A}BC$, where $\mathcal{A} \in S_A$ is the A-normal point of $\triangle UVW$.*

Remark 2.19. *Denote by R_A the intersection of the ray $x = 1$, $x \geq 0$ and S_A. Points of R_A correspond to right angle triangles. Points to the right and left of R_A correspond to, respectively, obtuse and acute triangles, see Figure 6.*

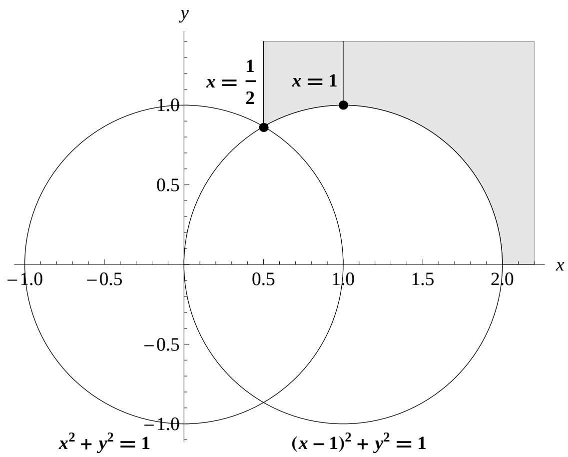

Fig.6. - the subdomains of S_A corresponding to obtuse and acute triangles.

Points on the intersection of the line $x = \frac{1}{2}$ and S_A correspond to isosceles acute triangles. Points on the intersection of the circle $(x-1)^2+y^2 = 1$ and S_A correspond to isosceles obtuse triangles. Points in the interior of S_A correspond to scalene triangles. The point $(\frac{1}{2}, \frac{\sqrt{3}}{2})$ corresponds to the equilateral triangle. Points on the intersection of the line $y = 0$ and S_A correspond to degenerate triangles excluding the similarity type with side lengths $0, c, c$, which corresponds to the point at infinity. In contrast to the C-vertex and B-vertex normal forms triangles for the A-vertex normal form are not bounded.

2.1.5. The circle normal form. Consider $\mathbb{R}^2$ with a Cartesian system of coordinates (x, y) and center O. We also consider polar coordinates $[r, \varphi]$ introduced in the standard way: the polar angle φ is measured from the positive x-axis going counterclockwise.

Note that angles α, β, γ of a nondegenerate triangle such that $\alpha \leq \beta \leq \gamma$ satisfy the system of inequalities

$$\begin{cases} 0 < \alpha \leq \frac{\pi}{3}, \\ \alpha \leq \beta \leq \frac{\pi-\alpha}{2}. \end{cases}$$

Similarity types of nondegenerate triangles are parametrized by one point in the domain in (α, β)-plane determined by the system

$$\begin{cases} \alpha > 0, \\ \beta \geq \alpha, \\ \beta \leq \frac{\pi}{2} - \frac{\alpha}{2}. \end{cases}$$

See Fig.7.

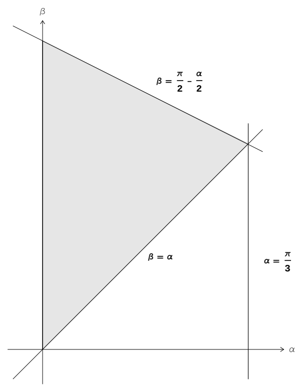

Fig.7. - parametrization of similarity types by (α, β).

For the normal form described in this subsection the vertex with the biggest angle will be fixed at $(1,0)$, to be consistent with previous notations we define $C = (1,0)$. For this normal form only nondegenerate triangles are considered.

In this case normal form triangles are inscribed in the unit circle $\mathbb{U} = \{x^2+y^2 = 1\}$ having C as one of the vertices.

Definition 2.20. *A triangle $\triangle ABC$ inscribed in $\mathbb{U}$ is called normal circle triangle* if

1. $0 \leq \alpha \leq \frac{\pi}{3}$,
2. $\alpha \leq \beta \leq \frac{\pi}{2} - \frac{\alpha}{2}$,
3. $C = (1,0)$,
4. the point A is above x-axis,
5. the point B is below x-axis.

Remark 2.21. *For a normal triangle $\triangle ABC$ we have that $\alpha \leq \beta \leq \gamma$.*

Remark 2.22. *A normal triangle with angles $\alpha \leq \beta \leq \gamma$ can be constructed in the following way:*

1. *choose a point B below the y-axis with the argument equal to 2α, where $0 \leq 2\alpha \leq \frac{2\pi}{3}$;*
2. *draw the bisector of $\angle BOC$, denote the intersection of this bisector with the arc BC having angle $2\pi - 2\alpha$ by D;*
3. *find the point $\widetilde{B}$ which is symmetric to B with respect to the x-axis;*
4. *choose a point A in the shorter arc $\widetilde{B}D$.*

See Figure 8.

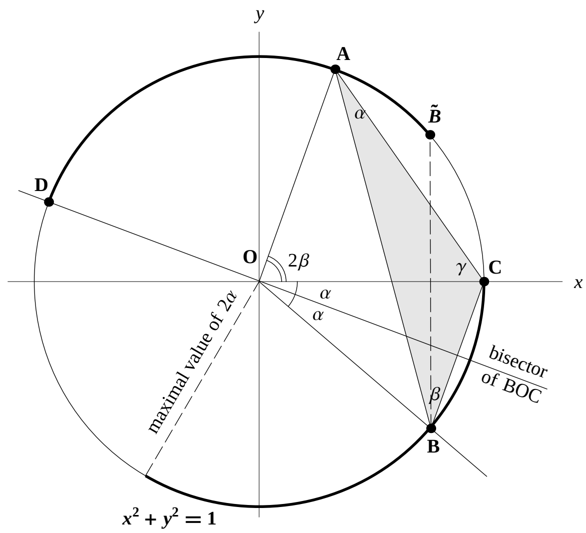

Fig.8. - construction of a normal circle triangle.

THEOREM 2.23. *For every nondegenerate triangle $\triangle UVW$ there exists a normal circle triangle $\triangle \mathcal{ABC}$ such that $\triangle UVW \sim \triangle \mathcal{ABC}$.*

Proof. Suppose $\triangle UVW$ has angles $\alpha \leq \beta \leq \gamma$. Let $\mathcal{B} \in \mathbb{U}$ be the point with polar coordinates $[1, -2\alpha]$. Let $\mathcal{A} \in \mathbb{U}$ be the point with polar coordinates $[1, 2\beta]$. Then since $\triangle \mathcal{ABC}$ is inscribed in $\mathbb{U}$ we have that $\angle \mathcal{BAC} = \alpha$, $\angle \mathcal{ABC} = \beta$ and thus $\triangle \mathcal{ABC} \sim \triangle UVW$.
□

THEOREM 2.24. *Let $\triangle A_1BC_1$ and A_2BC_2 be two distinct normal circle triangles: $A_1 \neq A_2$ or $B_1 \neq B_2$. Then $\triangle A_1BC_1 \not\sim \triangle A_2BC_2$.*

Proof. If $A_1 \neq A_2$, then $\angle B_1A_1C \neq \angle B_2A_2C$. The angle B_iA_iC is the smallest angle of $\triangle A_iB_iC$. We have that $\angle B_1A_1C \neq \angle B_2A_2C$ implies $\triangle A_1B_1C \not\sim \triangle A_2B_2C$.

If $B_1 \neq B_2$ and $A_1 = A_2$, then $\angle A_1CB_1 \neq \angle A_2CB_2$. The angle A_iCB_i is the largest angle of $\triangle A_iB_iC$. We have that $\angle A_1CB_1 \neq \angle A_2CB_2$ in this case implies $\triangle A_1B_1C \not\sim \triangle A_2B_2C$. □

REMARK 2.25. *The only isosceles normal triangles are normal triangles of type $\triangle B\widetilde{B}C$ and $\triangle BDC$. Right normal triangles are normal triangles with AB passing through O. Acute/obtuse normal triangles as normal triangles with O inside/outside $\triangle ABC$. In contrast to the one vertex normal forms triangles for the circle normal form are unbounded from below.*

REMARK 2.26. *Other normal forms of this type can be designed choosing another*

point instead of $(1,0)$ *and rearranging triangle points.*

2.1.6. Conversions.

DEFINITION 2.27. *Given a triangle with side lengths* a, b, c *define* $N_X(a,b,c)$ *to be the Cartesian plane coordinates of the* X*-normal point* ($X \in \{A,B,C\}$) *corresponding to this triangle. Note that* N_X *is a symmetric function. We can also think of arguments of* N_X *as multisets and think that* $N_X(a,b,c) = N_X(L)$, *where* L *is the multiset* $\{\{a,b,c\}\}$.

PROPOSITION 2.28. *Let* $\triangle ABC$ *has side lengths* $a \le b \le c$.

Then

1. $N_C(a,b,c) = \left(\frac{-a^2+b^2+c^2}{2c^2}, \frac{\sqrt{-a^4-b^4-c^4+2(a^2b^2+a^2c^2+b^2c^2)}}{2c^2}\right)$;
2. $N_B(a,b,c) = \left(\frac{-a^2+b^2+c^2}{2b^2}, \frac{\sqrt{-a^4-b^4-c^4+2(a^2b^2+a^2c^2+b^2c^2)}}{2b^2}\right)$;
3. $N_A(a,b,c) = \left(\frac{a^2-b^2+c^2}{2a^2}, \frac{\sqrt{-a^4-b^4-c^4+2(a^2b^2+a^2c^2+b^2c^2)}}{2a^2}\right)$.

Proof. 1. Translate, rotate and reflect $\triangle ABC$ so that $A = (0,0)$, $B = (c,0)$ and $C = (x,y)$ is in the first quadrant. For (x,y) we have the system

$$\begin{cases} x^2 + y^2 = b^2 \\ (c-x)^2 + y^2 = a^2 \end{cases}$$

and find

$$\begin{cases} x = \frac{-a^2+b^2+c^2}{2c} \\ y = \frac{\sqrt{-a^4-b^4-c^4+2(a^2b^2+a^2c^2+b^2c^2)}}{2c} \end{cases}$$

After the dilation by coefficient $\frac{1}{c}$ we get the given formula.

2. and 3. proved are in a similar way. □

PROPOSITION 2.29.

Let a triangle T *have angles* $\alpha \le \beta \le \gamma$.

Then

1. *its* C*-normal point is* $N_C(\frac{\sin\alpha}{\sin\gamma}, \frac{\sin\beta}{\sin\gamma}, 1)$;
2. *if* T *has the* C*-normal point* (x,y), *then it has angles* $\alpha = \arctan\frac{y}{x}$, $\beta = \arctan\frac{y}{1-x}$, $\gamma = \pi - \arctan\frac{y}{x} - \arctan\frac{y}{1-x}$.
3. *its* B*-normal point is* $N_B(\frac{\sin\alpha}{\sin\beta}, \frac{\sin\gamma}{\sin\beta}, 1)$;
4. *if* T *has the* B*-normal point* (x,y), *then it has angles* $\alpha = \arctan\frac{y}{x}$, $\beta = -\arctan\frac{y}{x} + \arctan\frac{y}{x-1}$, $\gamma = \pi - \arctan\frac{y}{x-1}$;

5. *its A-normal point is* $N_A(\frac{\sin\beta}{\sin\alpha}, \frac{\sin\gamma}{\sin\alpha}, 1)$*;*
6. *if* T *has the A-normal point* (x,y)*, then it has angles* $\alpha = -\arctan\frac{y}{x} + \arctan\frac{y}{x-1}$, $\beta = \arctan\frac{y}{x}$, $\gamma = \pi - \arctan\frac{y}{x-1}$.

Proof.

1. Let $\triangle ABC$ be the C-normal triangle with angles $\alpha \leq \beta \leq \gamma$, i.e. $|AB| = 1$. By the law of sines we have $b = |AC| = \frac{\sin\beta}{\sin\gamma}$ and $a = \frac{\sin\alpha}{\sin\gamma}$. By definition C has coordinates $N_C(\frac{\sin\alpha}{\sin\gamma}, \frac{\sin\beta}{\sin\gamma}, 1)$.

2. Let CD be a height of $\triangle ABC$. Formulas for angles are obtained considering $\triangle ACD$ and $\triangle BCD$.

3.,4.,5.,6. ar proved similarly. □

2.2. Normal forms of quadrilaterals. In this subsection we consider mutisets of 4 points in a plane. A multiset of 4 points can be interpreted as a quadrilateral. We exclude the case of one point of multiplicity 4. The multiset $Q = \{\{X, Y, Z, T\}\}$ is also denoted as $\square XYZT$. We define $Q_1 \sim Q_2$ provided there is an element of the dilation group g such that $g(Q_1) = Q_2$.

A set of 4 points defines a set of 6 distances between these points. Choosing any two points we can translate, rotate, reflect and dilate the given 4-point configuration so that the chosen two points have coordinates $A = (0,0)$ and $B = (1,0)$. Different normal forms can be obtained choosing pairs with different relative metric properties. In this paper we consider only the simplest case - two points having the maximal distance are mapped to the x-axis.

2.2.1. Longest distance normal form. Suppose we are given a quadrilateral $\square XYZT$ such that $|XY| \geq |XZ|$, $|XY| \geq |XT|$, $|XY| \geq |YZ|$, $|XY| \geq |YT|$, $|XY| \geq |ZT|$. We map X and Y by a dilation to the x-axis (to $A = (0,0)$ and $B = (1,0)$) and determine what are positions of the 2 remaining vertices C and D so that $\square XYZT \sim \square ABCD$.

Definition 2.30. *Let* $p \in \mathbb{R}^2$*. The mapping of* p *by reflections of* p *with respect to the x-axis and the line* $x = \frac{1}{2}$ *to the domain* $y \geq 0$, $x \geq \frac{1}{2}$ *is denoted by* p_s*.*

Definition 2.31. *Let* $p, p' \in \mathbb{R}^2$*. We say that* p *is quasilexicographically* smaller or equal to p', denoted by $p \lhd p'$, provided $p_s \prec p'_s$ or $p_s = p'_s$. Given two pairs $[p,q]$ and $[p',q']$ we define $[p,q] \lhd [p',q']$ provided $(p_s \prec p'_s)$ or $(p_s = p'_s$ and $q \lhd q')$.

Definition 2.32. *Let* $S_D(x_0, y_0) \subseteq \mathbb{R}^2$ *with* $(x_0, y_0) \in S_C$ *(for the definition of* S_C *see section 2.1.2) be the set of solutions of the following system of inequalities:*

$$
\left\{
\begin{array}{l}
x^2 + y^2 \le 1 \\
(x-1)^2 + y^2 \le 1 \\
(x - x_0)^2 + (y - y_0)^2 \le 1 \\
|x - \frac{1}{2}| \le |x_0 - \frac{1}{2}| \\
if\ |x - \frac{1}{2}| = |x_0 - \frac{1}{2}|, then\ |y| \le |y_0|
\end{array}
\right.
\tag{2.1}
$$

See Figure 9.

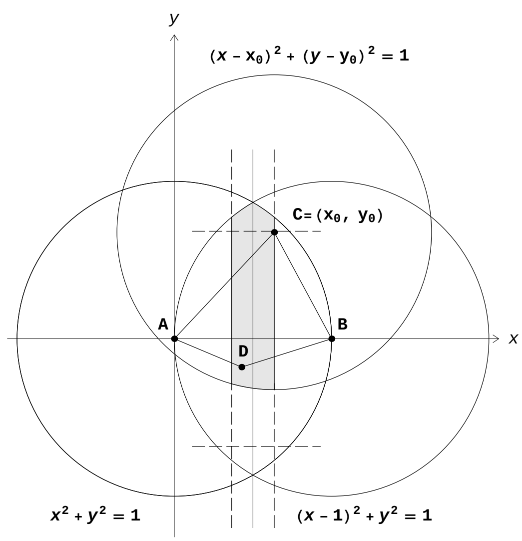

Fig.9. - example of the domain S_D.

REMARK 2.33. *Conditions for $p \in S_D(x_0, y_0)$ consist of two parts:*

1. *distance from p to A, B and (x_0, y_0) is less than or equal to 1;*
2. *$p_s \lhd (x_0, y_0)$.*

THEOREM 2.34. *Every $\square UVWZ$ (including multisets with multiplicities at most 3) in $\mathbb{R}^2$ is similar to $\square AB\mathcal{C}\mathcal{D}$, where $A = (0, 0)$, $B = (1, 0)$, $\mathcal{C} \in S_C$ and $\mathcal{D} \in S_D$.*

Proof. Let $UVWZ$ be a multiset of points in $\mathbb{R}^2$ with at least two distinct elements. Perform the following sequence of transformations:

1. translate and rotate the plane so that 2 points with the longest distance are on the x-axis, one vertex has coordinates $(0, 0)$ and another vertex has coordinates $(d, 0)$, $d > 0$; if there is more than one possibility to choose two points with the longest distance then choose this pair so that the remaining pair is the largest in the quasilexicographic order;

2. do the dilation with coefficient $\frac{1}{d}$, note that the vertices on the x-axis have coordinates $(0,0)$ and $(1,0)$, suppose the the other two vertices have coordinates (x_C, y_C) and (x_D, y_D);
3. if $|x_C - \frac{1}{2}| \neq |x_D - \frac{1}{2}|$, then put the point with the maximal $|x - \frac{1}{2}|$ value into S_C by reflections with respect to the x-axis and the line $x = \frac{1}{2}$;
4. if $|x_C - \frac{1}{2}| = |x_D - \frac{1}{2}|$, then put the point with the maximal value of $|y|$ into S_C by reflections with respect to the x-axis and the line $x = \frac{1}{2}$;
5. if $|x_C - \frac{1}{2}| = |x_D - \frac{1}{2}|$ and $|y_C| = |y_D|$, then map any of the points into S_C.

Denote the point which is mapped to S_C by this sequence of transformations by $C = (x_C, y_C)$ and the fourth point by $D = (x_D, y_D)$. For any $C = (x_0, y_0) \in S_C$ we have that $S_D(x_0, y_0) \neq \emptyset$.

We check that $D \in S_D(x_C, y_C)$. From conditions $|AD| \leq 1, |BD| \leq 1$, $|CD| \leq 1$ it follows that D satisfies the first three inequalities of the system 2.1. If $|y_D| > |y_C|$, then $|x_D - \frac{1}{2}| < |x_C - \frac{1}{2}|$ due to the quasilexicographic order condition. $\square$

Definition 2.35. *The longest distance normal form* of $\square UVWZ$ is $\square ABCD$ with $A = (0,0)$, $B = (1,0)$, $C = (x_C, y_C) \in S_C$ and $D \in S_D(x_C, y_C)$ constructed according to the algorithm given in the proof of 2.34.

Proposition 2.36. *Let $\square ABC_1D_1$ and $\square ABC_2D_2$ be two quadrilaterals constructed according to the longest distance normal form algorithm.*

If $C_1 \neq C_2$ or $D_1 \neq D_2$, then $\square ABC_1D_1 \not\sim \square ABC_2D_2$.

Proof. $D_i \lhd C_i$, therefore if $C_1 \neq C_2$, then $\square ABC_1D_1 \not\sim \square ABC_2D_2$.

Suppose $C_1 = C_2$ and $D_1 \neq D_2$. Under any similarity mapping C_1 must be mapped to C_2. If a similarity mapping fixes three noncollinear points A, B and C_i, then it must fix any other point of the plane. If A, B and C_i are on the x-axis, then D_i must also be on the x-axis and must be fixed. Therefore $D_1 \neq D_2$ implies $\square ABC_1D_1 \not\sim \square ABC_2D_2$ in this case. $\square$

Remark 2.37. *If $C_s = D_s$, then there are the following possibilities for the number of similarity types of quadrilaterals with a given $C \in S_C$: 1) 1 similarity type if $C = D$, 2) 2 similarity types if $C \neq D$ and $|AC| = |BC|$, or $C \neq D$ and C belongs to the x-axis or 3) 4 similarity types in other cases.*

3. Possible uses of normal forms in education. One vertex normal forms of triangles can be used to represent all similarity types of triangles in a single picture with all triangles having a fixed side, especially C-vertex and B-vertex normal forms. It may be useful to have an example for students showing that similarity type of triangle can be parametrized by coordinates of a single point. One vertex normal

forms can also be used in considering quadrilaterals.

The circle normal form of triangles may be useful teaching properties of circumscribed circles, e.g. inscribing triangles with given angles in a circle.

Normal forms of triangles can also be used to teach the idea of normal (canonical) objects using a case of simple and popular geometric constructions.

4. Conclusion and further development. It is relatively easy to define several normal forms of triangles up to similarity. Since the main purpose of this work is contribution to mathematics education only simplest approaches which may be used in teaching are considered in this paper. One approach is to map one side to the x-axis and use dilations and reflections to position the third vertex in a unique way, in this approach normal triangles are parametrized by one vertex. This approach can be generalized for quadrilaterals. Another approach considered in this paper is to design normal triangles as triangles inscribed in a unit circle. Further development in this direction may be related to using other figures related to a given triangle, for example, the inscribed circle, medians, altitudes or bisectors.

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